Acta Crystallographica Section A
Foundations of Crystallography
ISSN 0108-7673

Received 20 May 1999
Accepted 2 August 1999

Icosahedral quasiperiodic packing of fibres parallel to fivefold and threefold axes

Marc Audier[a] and Michel Duneau[b]*

[a]Laboratoire des Matériaux et du Génie Physique, CNRS UMR 5628, ENSPG, BP 46, 38402 Saint Martin d'Heres, France, and [b]Centre de Physique Théorique, Ecole Polytechnique, CNRS UMR 7644, 91128 Palaiseau, France. Correspondence e-mail: duneau@cpht.polytechnique.fr

Building rules are examined for an icosahedral quasiperiodic packing of fibres with axes parallel to the ten threefold axes, first employing an experimental construction and afterwards a mathematical demonstration using the cut-and-project method applied in hyperspace. As a result of this latter approach, very simple two-dimensional (2D) building rules are proposed. Similar simple 2D rules have also been proposed for the case of an icosahedral quasiperiodic packing with fibre axes parallel to the six fivefold axes [Duneau & Audier (1999). *Acta Cryst.* A**55**, 746–754]. Finally, the construction of another icosahedral quasiperiodic packing resulting from a combination of two groups of fibres respectively parallel to six fivefold and ten threefold axes is reported. A brief discussion is given on different particular mechanical behaviours which might *a priori* be expected from the various enantiomorphic properties of these packings.

1. Introduction

In a recent article, we reported rules for constructing the tightest quasiperiodic packing of fibres of the same circular cross section and parallel only to the six fivefold axes of the regular icosahedron (Duneau & Audier, 1999). Besides the description of an experimental construction, a mathematical method, related to the cut and projection of a periodic structure in a five-dimensional hyperspace, was proposed in order to determine unambiguously all the fibre positions in this icosahedral packing. As mentioned by Parkhouse & Kelly (1998), one of the aims of such research is to explore the feasibility of designing a fibrous composite that is both elastically isotropic and contains an appreciable volume fraction of reinforcement. But as the elasticity properties might, *a priori*, be improved as both the number of different fibre directions and the fibre volume fraction increase, we noticed that the best construction to consider is obtained when fibres are threaded parallel to the six fivefold and ten threefold axes of the icosahedral point-group symmetry *I*. Although an experimental construction of wood fibres threaded parallel to these 16 directions has been demonstrated, any description has not been given. Therefore, the main objective of the present work was to determine construction rules for this quasiperiodic packing of fibres. But since two subgroups of fibres parallel to either fivefold or threefold axes can be distinguished, the case of fibres threaded parallel only to threefold axes is firstly studied in this article. Both experimental and mathematical constructions of the closest packing are described. The characteristics of this first packing are analysed, in particular those of its two-dimensional (2D) projection along a threefold axis. Besides a formal mathematical method related to a description in hyperspace, we show that simple 2D building rules might as well be used in order to determine all the fibre positions. Similarly, for the packing of fibres parallel only to the fivefold icosahedral axes, 2D building rules are also shown to apply. Then the combination of both these subgroups of fibres parallel to fivefold and threefold axes is presented. Finally, a discussion summarizes various possibilities of tightest quasiperiodic packings of icosahedral symmetry when fibres of different diameters are used. With respect to mechanical properties, an influence of different combinations of enantiomorphic constructions is also conjectured.

2. Icosahedral packing of fibres parallel to threefold axes

2.1. Experimental construction

The first step of the construction is defined by a set of 30 fibres of circular cross section composed of ten equivalent subsets of three fibres parallel to the ten threefold axes of the icosahedral point-group symmetry *I*. The three first-step perspective views shown in Fig. 1(*a*) are related to this set, successively oriented along icosahedral axes of twofold, threefold and fivefold symmetry.

A useful projection to consider is perpendicular to a threefold axis because the construction can be reduced to a 2D problem from the subset of three fibres with axes perpendicular to the figure plane. As such a simplification has already been used in the case of the icosahedral packing of fibres parallel to fivefold axes, the other steps in the building procedure are therefore very similar to those previously reported (Duneau & Audier, 1999). Each subset of three parallel fibres defines at its intersection with a triangular face of the icosahedron the vertices of an equilateral triangle, the orientation of which is given by a 30° anticlockwise rotation from the vertices of the triangular face (Fig. 1*b*). Along each threefold icosahedral axis, the ten subsets of three parallel fibres exhibit three types of projection. With respect to the subset of three fibres represented in red in Fig. 1 and with axes

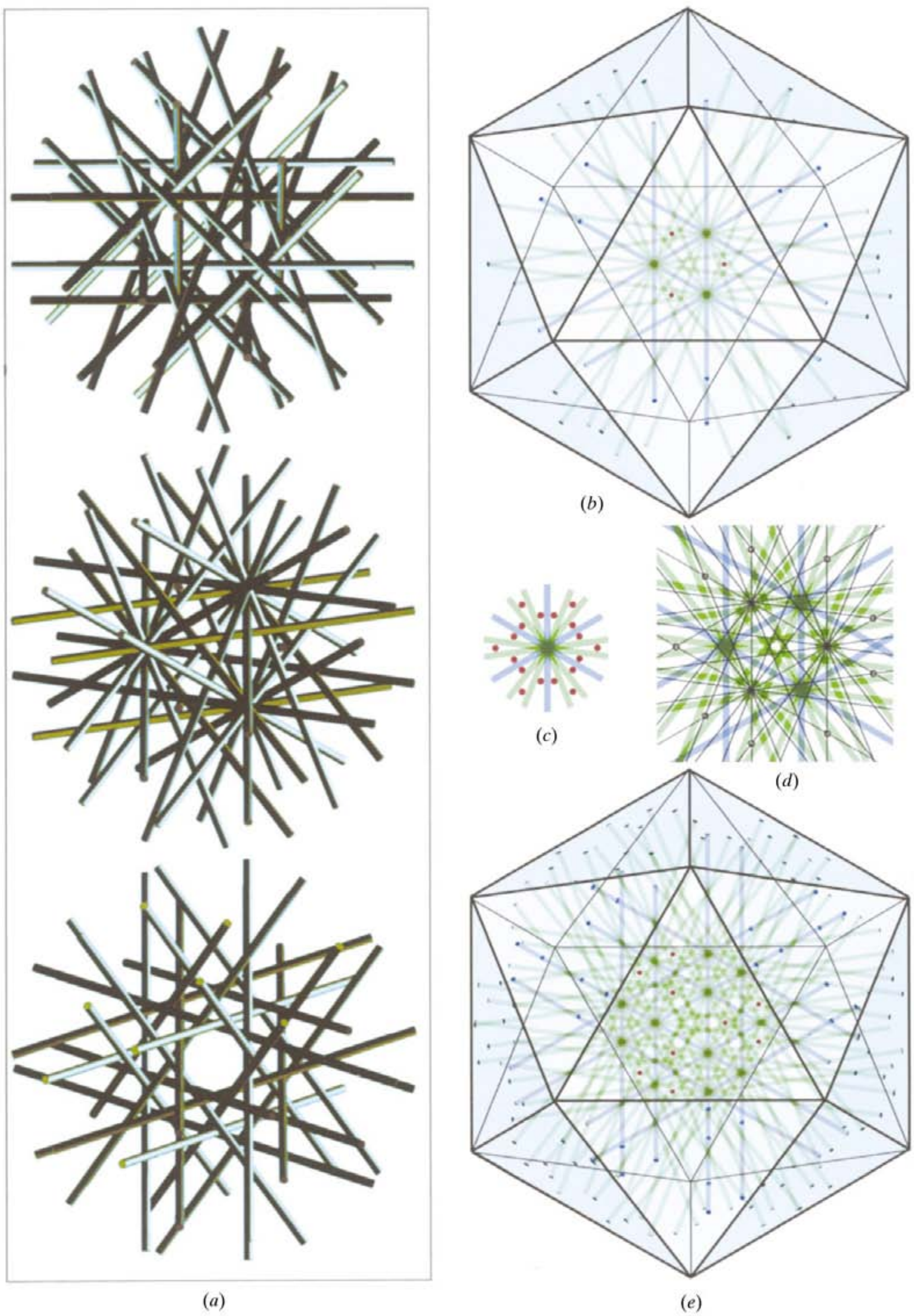


Figure 1
The first two steps for constructing the tightest icosahedral packing with fibres of the same circular cross section and with axes that are parallel only to the ten threefold icosahedral axes: (*a*) configuration of the first step viewed in perspective and successively along twofold, threefold and fivefold icosahedral directions; (*b*) first and (*e*) second building steps, showing that fibres can be threaded through holes drilled on the triangular facets of an icosahedron and according to criteria (*c*) and (*d*), explained in the text (§2.1). The different fibre colours correspond to different tilt angles of the fibre axes with respect to the normal to the figure plane.

that are perpendicular to the figure plane, there are three fibre subsets, in blue, with axes inclined at $2\tan^{-1}(\tau^{-2})$ ($\approx 41.81°$), and six other subsets, in green, with axes inclined at $\cos^{-1}(3^{-1})$ ($\approx 70.53°$). An enantiomorphic arrangement would be obtained by a 30° clockwise rotation. Both of these enantiomorphic packing orientations respect a closest-packing condition. For this first packing step, each fibre is in contact with four other fibres, while, for a rotation different from 30° but not equal to zero, each fibre would be in contact with two other fibres only and the size of the arrangement would be larger for the same fibre diameter. Note that no packing can be realized if the fibre axes belong to the mirror planes of the regular icosahedron (*i.e.* for $\pm k\pi/3$ rotations with k an integer). Therefore, as in the case of the icosahedral packing of fibres parallel to fivefold axes (Duneau & Audier, 1999), it is impossible to construct a packing of fibres with point-group symmetry equal to I_h.

To proceed to the packing of other fibres, three useful geometrical characteristics can be deduced from the threefold projection shown in Fig. 1(*b*). It turns out that:

(i) to the subset of three fibres that are normal to the plane of the figure (or vertical) corresponds, through a rotation of $\pm k\pi/3$ (with k an integer), a set of ninefold intersections defined by the projection of the other fibres (and conversely);

(ii) both symmetrical 2D patterns of vertical fibres and ninefold intersections must be entirely defined by rows of nodes parallel to the projection of nine threefold axes of the icosahedral point group (this second characteristic being rather obvious when considering that each ninefold intersection is precisely defined through the projection of nine fibres with axes parallel to nine different threefold icosahedral axes);

(iii) there are two minimal distances between a ninefold intersection and a vertical fibre (see Fig. 1*c*).

As a consequence, when adding new fibres, (*a*) the corresponding new ninefold intersections should overlap neither with previous intersections nor with a fibre normal to the plane of the figure (if not, fibres would be passing through the bulk of other fibres, which is excluded), and (*b*) their positions must correspond to the intersections of at least two rows of nodes along the projection of two threefold directions in the figure plane. Therefore, from the projection shown in Fig. 1(*b*), the positions of new vertical fibres were studied according to the following procedure:

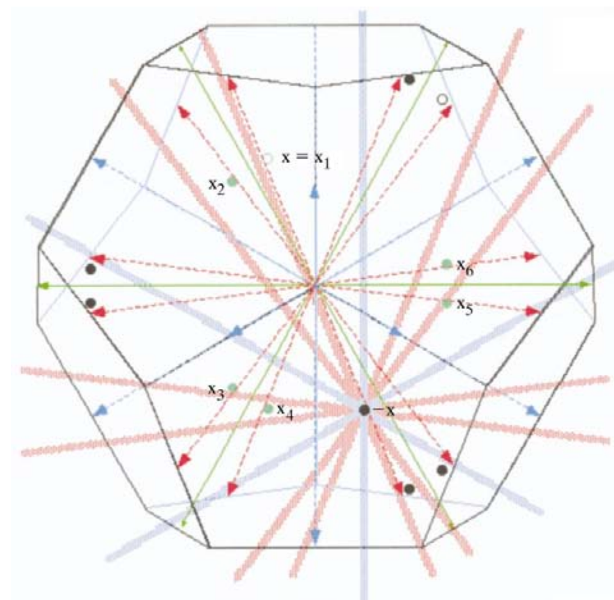
Nine lines parallel to the projection of nine threefold icosahedral axes were traced on drawings through the centre of each vertical fibre in order to determine where the different intersections of these lines correspond to holes of sufficient size to add a new fibre.

For instance, as shown on the enlarged pattern in Fig. 1(*d*), nine nearest possible positions, marked by empty circles, can be determined for the second step of the construction. However, through several tries, only six positions were finally retained for this second construction step (Fig. 1*e*).

Actually, such an experimental procedure was a little laborious and only a 2D pattern limited to about 90 nodes was determined and considered as a guide for researching a deterministic construction. Although the method hereafter developed requires learning some basic mathematics, let us mention now that very simple 2D building rules were finally deduced from an analysis of the very large 2D patterns that were obtained.

2.2. Mathematical demonstration of a deterministic construction

We first recall a few definitions regarding symmetry operations related to the icosahedral point group I which are



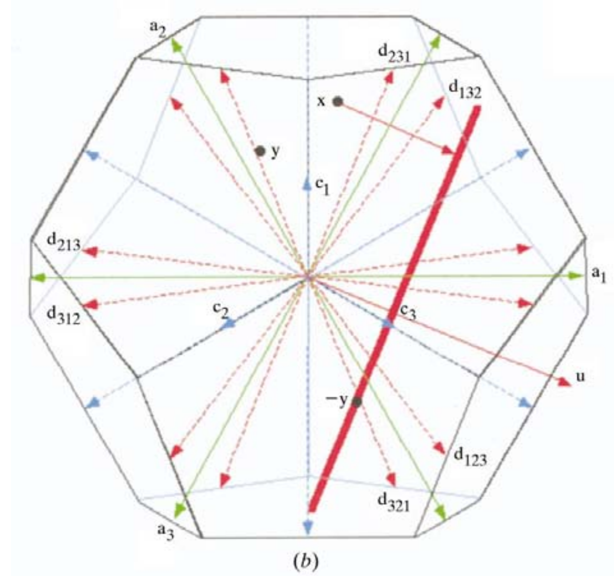


Figure 2
The projections of twofold axes and orbits of fibres onto a plane perpendicular to a threefold axis. (*a*) The three horizontal twofold axes are in green; nine oblique axes give three projections in blue and the six remaining give six other lines in red. The orbit ($\mathbf{x}$) of a vertical fibre at $\mathbf{x} = \mathbf{x}_1$ gives six vertical fibres at $\mathbf{x}_1,\ldots, \mathbf{x}_6$ and 54 oblique fibres, the projections of which build six stars centred at $-\mathbf{x}_i$. The star at $-\mathbf{x}$ contains 3 + 6 lines parallel to the blue and red projections of twofold axes. (*b*) The distances between the lines of two orbits ($\mathbf{x}$) and ($\mathbf{y}$) involve the distance along the direction $\mathbf{u}$ between $\mathbf{x}$ and the line at $-\mathbf{y}$ parallel to $\mathbf{d}_{231}$.

considered in a construction based on a hyperspace description. Properties of the resulting 2D pattern are later analysed and shown to be characteristic of simple 2D building rules.

2.2.1. Orbits of fibres. Let G denote a three-dimensional (3D) representation of the proper icosahedral group 235 of 60 elements. The subgroups that preserve a threefold axis are conjugate to the dihedral group $D_3 = 32$ of six elements. There are ten such axes and accordingly ten subgroups of G that are conjugate to D_3. The physical space is given a basis with three orthogonal axes x_1, x_2 and x_3. We assume that a threefold axis A of G is parallel to the x_3 axis and that a twofold axis of G is parallel to the x_1 axis. We denote H the dihedral subgroup of G which preserves this axis. The 15 twofold axes of G split into three axes which are perpendicular to A (the twofold rotations of H) and 12 axes which are oblique with respect to A.

The action of G on a line yields a generic orbit of 60 lines. We shall assume from now on that the lines do not run through the origin; otherwise intersections would occur when generating their orbit (this number is lowered to 30 if the line is invariant with respect to a twofold rotation). If a line is parallel to a given threefold axis, the orbit splits into ten subsets, each one containing six (or three, respectively) lines parallel to a particular threefold axis. We consider now the projection of the whole orbit onto the horizontal (x_1, x_2) plane P, perpendicular to the threefold axis A, which runs through the origin (Fig. 2*a*). We first obtain six points $\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(6)}\}$ in P corresponding to the subset of six lines parallel to A. These points build an orbit of the dihedral group H. As mentioned previously, the projection of the remaining 54 oblique lines gives six stars $\{\mathbf{S}^{(1)}, \ldots, \mathbf{S}^{(6)}\}$ of nine lines each, centred on the points $\{-\mathbf{x}^{(1)}, \ldots, -\mathbf{x}^{(6)}\}$. The star $\mathbf{S}^{(i)}$ centred at $-\mathbf{x}^{(i)}$ ($i = 1, \ldots, 6$) is the projection on P of the oblique lines of the orbit, which are obtained from the vertical line at $\mathbf{x}^{(i)}$ by twofold rotations with oblique axes. It follows from elementary geometry that the directions of the stars are parallel to the projections of these oblique twofold axes.

2.2.2. Construction by the cut-and-project method. We come now to the six-dimentional (6D) indexing of vectors and positions of lines. The geometrical framework is related to that of the 3D Penrose tiling, where nodes are the 3D 'parallel' projections of points in a 6D cubic lattice Λ generated by a standard orthogonal basis $\{\varepsilon_1, \ldots, \varepsilon_6\}$ (Janot, 1994; Duneau, 1994*b*). The parallel projection L of Λ is the $\mathbf{Z}$ module generated by the icosahedral basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_6\}$, where the vectors point to six vertices of an icosahedron. The 3D 'perpendicular' projection L^* of Λ is the $\mathbf{Z}$ module spanned by a basis $\{\mathbf{e}_1^*, \ldots, \mathbf{e}_6^*\}$. Therefore, a point $\boldsymbol{\xi} = \sum x_i \varepsilon_i$ in the 6D space gives the parallel projection $\mathbf{x} = \sum x_i \mathbf{e}_i$ and the perpendicular projection $\mathbf{x}^* = \sum x_i\, \mathbf{e}_i^*$.

Let $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ denote the three vectors of L parallel to the twofold axes lying in the plane P and of norm $a = (2 + 2/5^{1/2})^{1/2}|\mathbf{e}_1|$. We assume that $\mathbf{a}_1$ is parallel to the x_1 axis and that $\mathbf{a}_2$ and $\mathbf{a}_3$ have relative orientations $2\pi/3$ and $4\pi/3$ with respect to $\mathbf{a}_1$. The three vectors $\mathbf{b}_i = \tau^{-1}\mathbf{a}_i$ also belong to L and we have $\sum \mathbf{a}_i = \sum \mathbf{b}_i = 0$. The $\mathbf{Z}$ module $L_P = L \cap P$ is generated by $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ but it is only of rank 4. Any point $\mathbf{x}$ of L_P can be written $\mathbf{x} = \sum (m_i\mathbf{a}_i + n_i\mathbf{b}_i)$, where the coordinates m_i and n_i are integers. These coordinates can be chosen in order that $\underline{m} = \sum m_i$ and $\underline{n} = \sum n_i$ are both in the set $\{-1, 0, 1\}$.

In perpendicular space, we have the corresponding vectors $\mathbf{a}_i^*$ and $\mathbf{b}_i^*$ ($i = 1, 2, 3$) with the relationships $\mathbf{b}_i^* = -\tau\mathbf{a}_i^*$ and $|\mathbf{b}_i^*| = a$. If $\mathbf{x}$ lies in L_P, the perpendicular image $\mathbf{x}^*$ belongs to the $\mathbf{Z}$ module $L_P^* = L^* \cap P^*$ and we have $\mathbf{x}^* = \sum (m_i\mathbf{a}_i^* + n_i\mathbf{b}_i^*)$.

Both L_P and L_P^* contain natural $\mathbf{Z}$ modules $L_{P,0,0}$ and $L_{P,0,0}^*$ defined by the condition $\underline{m} = \underline{n} = 0$. L_P and L_P^* also contain the translations $L_{P,\mu,\nu}$, $L_{P,\mu,\nu}^*$ for $\mu = -1, 0, 1$, and $\nu = -1, 0, 1$, which correspond to points $\mathbf{x}$ and $\mathbf{x}^*$ such that $\underline{m} = \mu$ and $\underline{n} = \nu$.

The relationships $\mathbf{b}_i = \tau^{-1}\mathbf{a}_i$ and $\mathbf{b}_i^* = -\tau\mathbf{a}_i^*$ imply that L_P and L_P^* are invariant with respect to an inflation T defined in L_P by $T(\mathbf{a}_i) = \tau\mathbf{a}_i = \mathbf{a}_i + \mathbf{b}_i$, $T(\mathbf{b}_i) = \tau\mathbf{b}_i = \mathbf{a}_i$, and in L_P^* by $T(\mathbf{a}_i^*) = -\tau^{-1}\mathbf{a}_i^* = \mathbf{a}_i^* + \mathbf{b}_i^*$ and $T(\mathbf{b}_i^*) = -\tau^{-1}\mathbf{b}_i^* = \mathbf{a}_i^*$. One can easily check that $L_{P,0,0}$ and $L_{P,0,0}^*$ are also T-invariant and that $L_{P,\mu,\nu}$ and $L_{P,\mu,\nu}^*$ build orbits of length 8. More precisely, $TL_{P,\mu,\nu} = L_{P,\mu+\nu,\mu}$, where the indices can be reduced modulo 3 in $\{-1, 0, 1\}$, and $T^8L_{P,\mu,\nu} = L_{P,\mu,\nu}$.

Besides $\mathbf{a}_2$ and $\mathbf{a}_3$, the orbit of $\mathbf{a}_1$ contains 12 more vectors, which are parallel to different oblique twofold axes. The projection in P of these oblique twofold axes only yields nine different directions (Fig. 2*b*): three of them are parallel to $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$, and perpendicular to $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$, respectively. The six remaining are parallel to the vectors $\mathbf{d}_{ijk} = \mathbf{a}_i - \mathbf{a}_j + \tau^{-1}(\mathbf{a}_k - \mathbf{a}_j)$, where $i \neq j \neq k \in \{1, 2, 3\}$.

The horizontal twofold rotations with axes parallel to $\mathbf{a}_i$ ($i = 1, 2, 3$) transform a vertical line into a vertical line. Each oblique twofold rotation transforms a vertical line at $\mathbf{x}$ into an oblique line. The projection in P of this line is parallel to the projection of the twofold axis and it runs through the point $-\mathbf{x}$. Finally, the nine directions $\mathbf{c}_i$ and $\mathbf{d}_{ijk}$ split into two orbits of the dihedral group H.

The set of vertical fibres will be constructed by a cut method restricted to the parallel plane P and the perpendicular plane P^* in which a window W will be defined. More precisely, a vertical fibre will be located at $\mathbf{x}$ if and only if $\mathbf{x}^*$ falls inside W. The overall icosahedral symmetry requires that the set of vertical fibres is invariant with respect to H, which amounts to requiring the dihedral symmetry of the window W.

2.2.3. Minimal distances. Our purpose is now to specify a set of vertical fibres by the set S of their intersections in the plane P with the following constraints: (i) S is invariant with respect to the dihedral group H, (ii) the orbit of the vertical fibres specified by S satisfies a condition of minimal distance. Among all possible solutions, we wish to find those that maximize the relative volume of fibres of which the common diameter is given by the shortest distance between them.

Assume that two vertical fibres are located at $\mathbf{x}$ and $\mathbf{y}$ in the plane P. Each one generates an orbit. Owing to the icosahedral symmetry, the minimal distance between these fibres can be evaluated by considering the vertical fibre at $\mathbf{x}$ and the orbit of the fibre at $\mathbf{y}$. In the plane P, the projection of this orbit consists of six points $\{\mathbf{y}^{(1)} = \mathbf{y}, \mathbf{y}^{(2)}, \ldots, \mathbf{y}^{(6)}\}$ and six stars centred at $-\mathbf{y}^{(i)}$ ($i = 1, \ldots, 6$). It will turn out that the minimal distance between the vertical fibre at $\mathbf{x}$ and the orbit of the fibre at $\mathbf{y}$ always occurs between $\mathbf{x}$ and one of the six stars at

$-\mathbf{y}^{(i)}$ ($i = 1, \ldots, 6$). Thanks to the assumed symmetry of the construction, we only have to consider the distance between $\mathbf{x}$ and the star $\mathbf{S}$ centred at $-\mathbf{y}$. But since the nine directions of the stars belong to two orbits of H, we merely have to examine two distances:

(i) The distance δ_1 between $\mathbf{x}$ and the line of the star $\mathbf{S}$ which is parallel to $\mathbf{c}_1$. This distance is given by

$$\delta_1 = (\mathbf{x} + \mathbf{y}) \cdot \mathbf{a}_1 / a,$$

where $a = |\mathbf{a}_1|$.

We define the functions $F(\mathbf{x}) = 2m_1 - m_2 - m_3$ and $G(\mathbf{x}) = 2n_1 - n_2 - n_3$. Then we have

$$\delta_1 = [F(\mathbf{z}) + G(\mathbf{z})/\tau] a/2,$$

where $\mathbf{z} = \mathbf{x} + \mathbf{y}$.

If $\mathbf{x} = \sum (m_i \mathbf{a}_i + n_i \mathbf{b}_i)$ and $\mathbf{y} = \sum (p_i \mathbf{a}_i + q_i \mathbf{b}_i)$, we have $\mathbf{z} = \sum (r_i \mathbf{a}_i + s_i \mathbf{b}_i)$ with $r_i = m_i + p_i$ and $s_i = n_i + q_i$. We can check that $F(\mathbf{z}) = -\underline{r} \pmod 3$ and $G(\mathbf{z}) = -\underline{s} \pmod 3$ with $\underline{r} = \sum r_i$ and and $\underline{s} = \sum s_i$.

(ii) The distance $\delta_{\mathbf{u}}$ between $\mathbf{x}$ and the line of the star $\mathbf{S}$ which is parallel to $\mathbf{d}_{231}$. This distance is given by

$$\delta_{\mathbf{u}} = (\mathbf{x} + \mathbf{y}) \cdot \mathbf{u} / (2^{1/2} a),$$

where $\mathbf{u} = \mathbf{a}_1 - \mathbf{b}_2 = \mathbf{a}_1 - \mathbf{a}_2/\tau$ is orthogonal to $\mathbf{d}_{231} = \mathbf{a}_2 - \mathbf{a}_3 + \tau^{-1}(\mathbf{a}_1 - \mathbf{a}_3)$ and has the norm $|\mathbf{u}| = 2^{1/2} a$. We define the functions $H(\mathbf{x}) = 2m_1 - m_2 - m_3 + n_1 - 2n_2 - n_3$ and $K(\mathbf{x}) = m_1 - 2m_2 + m_3 + n_1 + n_2 - 2n_3$. Then we have

$$\delta_{\mathbf{u}} = [H(\mathbf{z}) + K(\mathbf{z})/\tau] a / (2 \times 2^{1/2}).$$

One can check that $H(\mathbf{z}) = -\underline{r} + \underline{s} \pmod 3$ and $K(\mathbf{z}) = \underline{r} + \underline{s} \pmod 3$.

All other distances will be taken into account when considering other points of the H orbit $\{\mathbf{x}^{(1)} = \mathbf{x}, \ldots, \mathbf{x}^{(6)}\}$ of $\mathbf{x}$ and other stars centred at points $-\mathbf{y}^{(i)}$.

The scalar products $\mathbf{z} \cdot \mathbf{a}_1$ and $\mathbf{z} \cdot \mathbf{u}$ are related to the corresponding ones $\mathbf{z}^* \cdot \mathbf{a}_1^*$ and $\mathbf{z}^* \cdot \mathbf{u}^*$ in perpendicular space where $\mathbf{z}^* = \sum (r_i \mathbf{a}_i^* + s_i \mathbf{b}_i^*)$. For simplicity, we shall identify the two planes by letting $\mathbf{a}_i^* = \mathbf{b}_i$ and $\mathbf{b}_i^* = -\mathbf{a}_i$. Then

$$\mathbf{z}^* \cdot \mathbf{a}_1^* = [-G(\mathbf{z}) + F(\mathbf{z})/\tau] a^2 / (2\tau),$$
$$\mathbf{z}^* \cdot \mathbf{u}^* = [-K(\mathbf{z}) + H(\mathbf{z})/\tau] a^2 / (2\tau^2).$$

Thus, we see that, up to scaling factors, $\mathbf{z} \cdot \mathbf{a}_1$ and $\mathbf{z}^* \cdot \mathbf{a}_1^*$ on the one hand, and $\mathbf{z} \cdot \mathbf{u}$ and $\mathbf{z}^* \cdot \mathbf{u}^*$ on the other hand, are related through the conjugation $\tau \to -1/\tau$. More precisely, we can interpret $F(\mathbf{z}) + G(\mathbf{z})/\tau$ and $G(\mathbf{z}) - F(\mathbf{z})/\tau$ as the 'parallel' and 'perpendicular' projections of a 2D lattice point $[F(\mathbf{z}), G(\mathbf{z})]$ on the lines of slope τ^{-1} and $-\tau$, respectively. Similarly, $H(\mathbf{z}) + K(\mathbf{z})/\tau$ and $K(\mathbf{z}) - H(\mathbf{z})/\tau$ can be seen as the projections of the 2D lattice point $[H(\mathbf{z}), K(\mathbf{z})]$ on the same directions.

We now come back to the question of minimal distances between fibres. The required minimal bounds on the distances can be handled by considering the size of the window that specifies the selected lattice points. In view of the above discussion, we shall obtain lower bounds on the distances by specifying conditions in perpendicular space such as $c_1 < G(\mathbf{z}) - F(\mathbf{z})/\tau < C_1$ and $c_{\mathbf{u}} < K(\mathbf{z}) - H(\mathbf{z})/\tau < C_{\mathbf{u}}$, where c_1, C_1, $c_{\mathbf{u}}$ and $C_{\mathbf{u}}$ are constants to be specified. These conditions on the sum $\mathbf{z} = \mathbf{x} + \mathbf{y}$ lead to the conditions $c_1/2 < G(\mathbf{x}) - F(\mathbf{x})/\tau < C_1/2$, and $c_{\mathbf{u}}/2 < K(\mathbf{x}) - H(\mathbf{x})/\tau < C_{\mathbf{u}}/2$ on $\mathbf{x}$, and similar conditions on $\mathbf{y}$. Of course, each condition must be completed with respect to the symmetry of the dihedral group in such a way that the whole structure shows the required symmetry.

If $\mathbf{x}$ and $\mathbf{y}$ belong to the subset $L_{P,\mu,\nu}$, then $\mathbf{z} = \mathbf{x} + \mathbf{y}$ belongs to $L_{P,2\mu,2\nu}$ and we have $\underline{r} = 2\mu \pmod 3$ and $\underline{s} = 2\nu \pmod 3$. Therefore, $F(\mathbf{z}) = -2\mu \pmod 3$, $G(\mathbf{z}) = -2\nu \pmod 3$, $H(\mathbf{z}) = 2(\nu - \mu) \pmod 3$ and $K(\mathbf{z}) = 2(\mu + \nu) \pmod 3$. This means that the lattice points $[F(\mathbf{z}), G(\mathbf{z})]$ and $[H(\mathbf{z}), K(\mathbf{z})]$ run over (translated) sublattices of index 9 in $\mathbf{Z}^2$.

The case $\mu = \nu = 0$ can be rejected at once since the $\mathbf{Z}$ module $L_{P,0,0}$ is symmetric with respect to the origin. This implies that for any reasonable window we could select pairs of points $\mathbf{x}$ and $\mathbf{y} = -\mathbf{x}$ so that $\mathbf{z} = 0$, entailing an intersection of fibres.

In view of the T-inflation property of the construction, we can examine only one case among the eight remaining ones, say $\mu = 1$, $\nu = 0$, so that the points will be selected in $L_{P,1,0}$. It follows that $F(\mathbf{z}) = 1 \pmod 3$, $G(\mathbf{z}) = 0 \pmod 3$, $H(\mathbf{z}) = 1 \pmod 3$ and $K(\mathbf{z}) = -1 \pmod 3$. If Δ is the diameter of the fibres, the condition for non intersection reads $\delta_1 \geq \Delta$ and $\delta_{\mathbf{u}} \geq \Delta$ for all selected points and similar inequalities deduced from the action of the dihedral group. This is equivalent to the following conditions:

$$|F(\mathbf{z}) + G(\mathbf{z})/\tau| \geq 2\Delta/a$$

and

$$|H(\mathbf{z}) + K(\mathbf{z})/\tau| \geq 2 \times 2^{1/2} \Delta/a.$$

The τ^8-inflation symmetry of $L_{P,1,0}$ allows the distances Δ in an interval of type $(D, \tau^8 D)$ to be sought so that a finite number of solutions can be calculated. We have retained the following one which gives the most compact form:

(*a*)

$$|F(\mathbf{z}) + G(\mathbf{z})/\tau| \geq 3\tau - 4,$$

which implies $\delta_1 \geq (3\tau - 4)a/2 \approx 0.427051a$;

(*b*)

$$|H(\mathbf{z}) + K(\mathbf{z})/\tau| \geq 2 - \tau,$$

which implies $\delta_{\mathbf{u}} \geq (2 - \tau)a/(2 \times 2^{1/2}) \approx 0.135045a$.

The maximal diameter of the fibres is therefore $\Delta = \mathrm{Inf}\{\delta_1, \delta_{\mathbf{u}}\} = (2 - \tau)a/(2 \times 2^{1/2})$ and consequently all contacts between fibres occur along directions of type $\mathbf{u}$.

These bounds are satisfied by the following conditions in perpendicular space [the constraints on $F \pmod 3$, $G \pmod 3$, $H \pmod 3$ and $K \pmod 3$ are crucial for this result]:

(*a*)

$$-(2 + 4\tau) < G(\mathbf{z}) - F(\mathbf{z})/\tau < 1 + 2\tau;$$

(*b*)

$$-(3 + 4\tau) < K(\mathbf{z}) - H(\mathbf{z})/\tau < 3 + 5\tau.$$

As mentioned earlier, such conditions must be completed with respect to the dihedral group. In perpendicular space, these bounds specify a set given by an intersection of equilateral triangles of different orientations and sizes. It turns out that the symmetric set deduced from bounds (a) and (b) is actually a triangle defined by the three conditions of type $G(\mathbf{z}) - F(\mathbf{z})/\tau = 1 + 2\tau$. Finally, since these conditions bear on the sum $\mathbf{z} = \mathbf{x} + \mathbf{y}$, we conclude that the window W which specifies the points $\mathbf{x}$ is defined by the three bounds of type

$$G(\mathbf{x}) - F(\mathbf{x})/\tau < (1 + 2\tau)/2.$$

The selected points must satisfy the further conditions $F(\mathbf{x}) = -\,(\mathrm{mod}\,3)$ and $G(\mathbf{x}) = 0\,(\mathrm{mod}\,3)$ which characterize $L_{P,1,0}$.

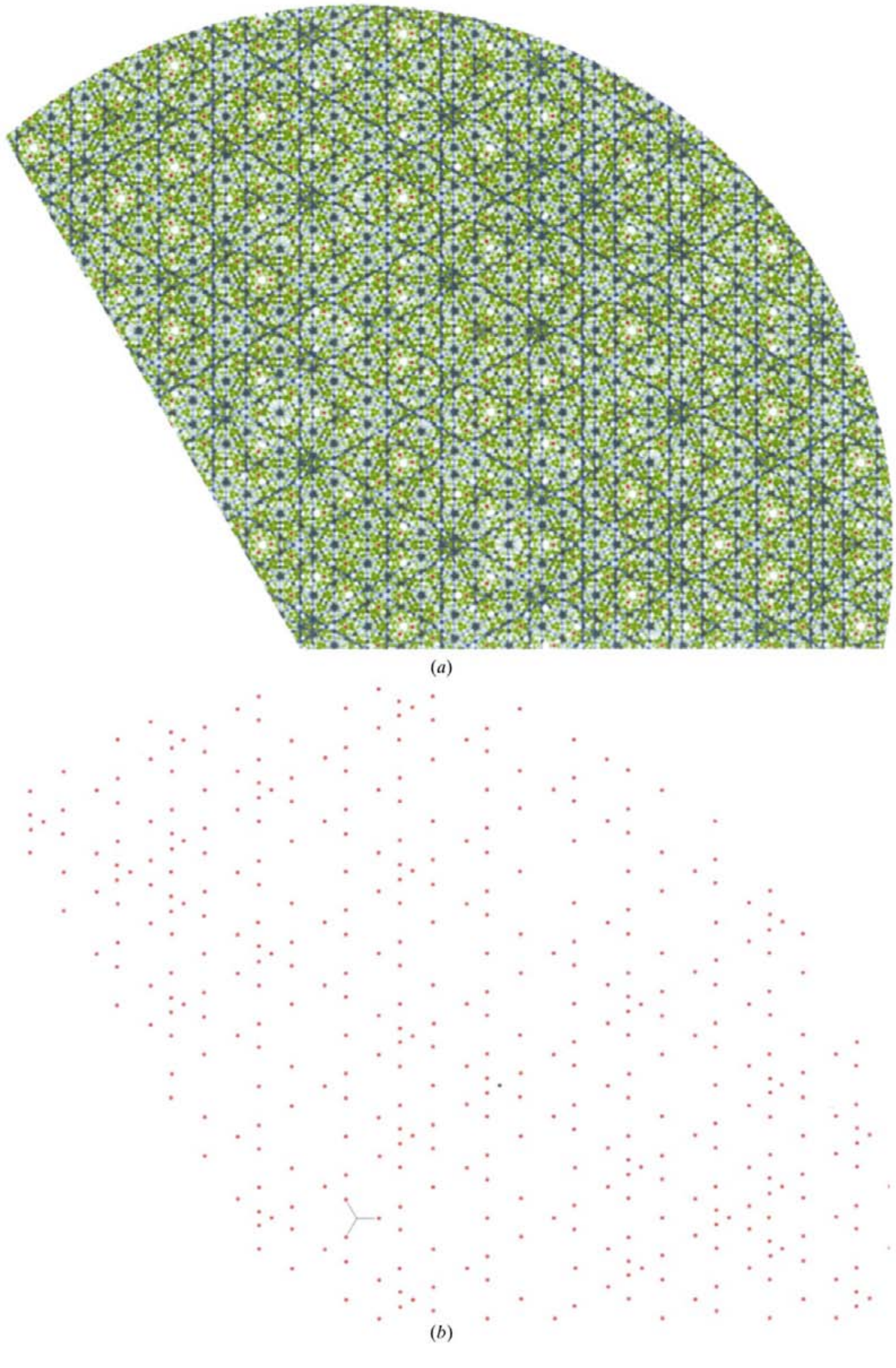


Figure 3
Threefold projection and corresponding 2D pattern related to the packing of fibres parallel to the ten threefold icosahedral axes.

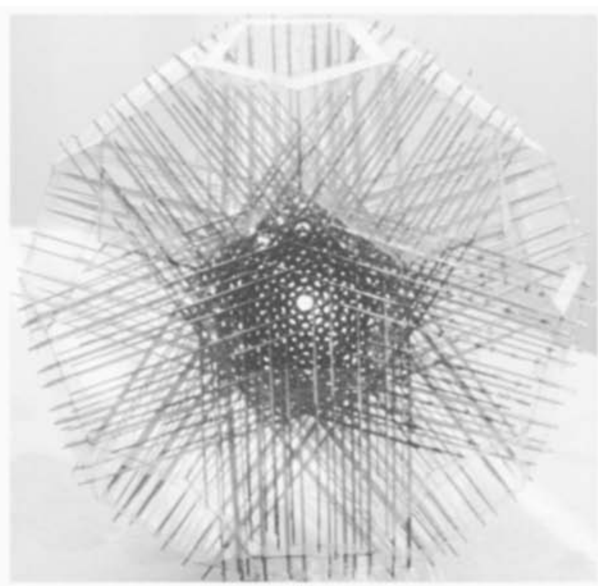
Patterns as shown in Fig. 3 were computed using the above formalism. The first pattern (a) corresponds to a threefold projection of the packing and the second one (b) to the subset of fibres, which are perpendicular to the figure plane. The different colours of fibre are related to their tilt angle, as in Figs. 1(b) and 1(e). It can be pointed out that the vertical fibres are only in contact with the six subsets of green fibres [*i.e.* inclined at $\cos^{-1}(3^{-1}) \approx 70.53^\circ$ from the normal]. Besides, there are two types of holes, the sizes of which are larger than the fibre diameter. The lattice associated with the largest holes corresponds to a self-similar subset of vertical fibres, through a τ^3 inflation and a rotation of $\pm k\pi/3$ (k is an integer). Therefore, it can easily be deduced from Fig. 3(a) that fibres of diameter τ times larger can be threaded through such holes and according to the same icosahedral symmetry. These fibres cannot be in contact with one another but only with the six subsets of green fibres.

Characteristics of the 2D pattern formed by the vertical fibres (Fig. 3b) will be examined in §2.2.5 in relation to the generation of the pattern from simple 2D building rules.

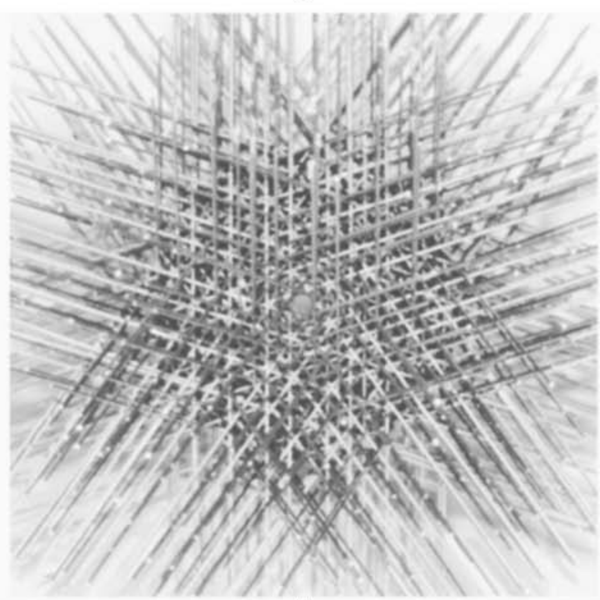
A model of this icosahedral packing was fabricated with 570 stainless steel rods, each of 1.2 mm diameter, in order to prove its feasibility. The two photographs in Fig. 4 show the construction in its plastic support and a view along a fivefold direction after the removal of the support. This view can be compared with the fivefold perspective view of the first set of 30 fibres (Fig. 1a) and with the fivefold projection shown in Fig. 7(c).

2.2.4. Density of the packing. The above construction involves the two planes P and P^* and the two $\mathbf{Z}$ modules L_P and L_P^*. The conditions $\sum \mathbf{a}_i = 0$ and $\sum \mathbf{b}_i = 0$ show that we actually use a four-dimensional (4D) lattice, Λ_4, which can be spanned by the basis $\{(\mathbf{a}_1, \mathbf{a}_1^*), (\mathbf{a}_2, \mathbf{a}_2^*), (\mathbf{b}_1, \mathbf{b}_1^*), (\mathbf{b}_2, \mathbf{b}_2^*)\}$ in the 4D space $P \times P^*$. The volume of the unit cell can be evaluated to $\omega = 15/4\tau^{-2}a^4 \approx 1.42337a^4$. The conditions $\mu = 1$ and $\nu = 0$ select a sublattice of index 9 in Λ_4, the unit cell of which has volume 9ω. The area of the triangular window W can be calculated using the relationship $\mathbf{x}^* \cdot \mathbf{a}_1^* = [-G(\mathbf{x}) + H(\mathbf{x})/\tau]a^2/(2\tau)$ and the bounds $H(\mathbf{x}) - G(\mathbf{x})/\tau < (1 + 2\tau)/2$. We obtain $|W| = 3^{1/2}3\tau^6a^2/16 \approx 5.827\,57a^2$. According to a general argument of the cut-and-project method, the density of points in P is equal to $|W|/(9\omega) = \tau^8/(3^{1/2}60)/a^2 \approx 0.452\,053/a^2$. The diameter of the fibres was given by $\Delta = (2 - \tau)a/(2^{1/2}2)$; therefore, the relative volume occupied by fibres parallel to a given A_3 axis is $\pi\Delta^2|W|/(36\omega) = \pi\tau^4/(3^{1/2}1920) \approx 0.006\,474\,9$. Finally, the relative volume occupied by all fibres is 10 times larger, *i.e.* $v = \pi\tau^4/(192 \times 3^{1/2}) \approx 0.064\,749$.

2.2.5. Inflation properties and covering clusters. The vertical fibres are located on the set S of points $\mathbf{x}$ of $L_{P,1,0}$ selected by the window W defined above. W is an equilateral triangle defined by the three conditions $\mathbf{x}^* \cdot \mathbf{a}_i^* > -\tau^2a^2/2$, the vertices of which are the points $-\tau^3/2\mathbf{b}_i^* = \tau^3/2\mathbf{a}_i$. If we apply the transformation $-T^3$ to this construction, we obtain the set $S' = -\tau^3 S$ in P, the points of which belong to $-T^3L_{P,1,0} = L_{P,0,1}$. S' is defined by the window $W' = \tau^{-3}W$ in perpendicular space, a triangle with vertices $-\mathbf{b}_i^*/2$. We show now that there exists a finite cluster C such that $C + S' = S$. In other words, the union of copies of C centred on the set S' covers all points of S. Such a property of quasiperiodic structures was first presented by Gummelt (1994) and Duneau (1994a). This is of course reminiscent of the property of the unit cell for a periodic structure. In quasiperiodic structures, however, the clusters have to overlap partially in order to cover all points. Translated in perpendicular space, the covering property means that $C^* + W' = W$, where C^* is the perpendicular image of the cluster. If $C = \{\mathbf{c}_1, \ldots, \mathbf{c}_k\}$, this is equivalent to $\cup_{i=1,\ldots,k}(W' + \mathbf{c}_i^*) = W$. As was proved by Duneau (1994a), this is possible only for windows W with particular shapes (excluding, for instance,



(a)



(b)

Figure 4

Photographs of a model of icosahedral packing of fibres with axes that are parallel to the ten threefold icosahedral axes. (a) The model along a fivefold direction in its plastic support; (b) the same view after removal of the plastic support. (Note: a few fibres were stuck together before removing the support.)

circular windows). Since W and W' select points of $L_{P,1,0}$ and $L_{P,0,1}$, respectively, we see that the vertices $\mathbf{c}_i$ of the cluster must belong to $L_{P,1,0} - L_{P,0,1} = L_{P,1,-1}$. A covering cluster that satisfies all these requirements is shown in Fig. 5. In this figure, the subset of fibres perpendicular to the figure plane (as already shown in Fig. 3) and its related $-\tau^3$-inflated subset are respectively represented by red and green nodes. The smallest cluster (see the small insert in Fig. 5) allowing one to cover all the red nodes is defined by six discs of orange colour. Then a large cluster centred on a disc of violet colour and defined by an arrangement of six small clusters (see the largest insert) can in turn be used to cover both the red and the green nodes. Therefore, the lattice construction becomes very simple using this large cluster and knowing that both red- and green-node sublattices are related by $-\tau^3$ inflation. The beginning of the construction requires the determination of an arrangement of at least three clusters in order to respect a threefold symmetry. On account of the $-\tau^3$ inflation property, these three clusters must be centred at the three vertices of an equilateral triangle such that three orange discs define the vertices of a small equilateral triangle in the middle of the previous one through a twofold rotation and with a τ^3 size ratio. There is only one solution and one obtains a pattern formed by 102 orange discs and 3 violet discs (which superimpose to red and green nodes, respectively). Then a new part of the pattern of green nodes can be determined by $-\tau^3$ inflation from the positions of the 102 orange discs. Other large clusters must be centred on these new green nodes and the same procedure has to be repeated recurrently.

3. Icosahedral packing of fibres parallel to threefold and fivefold axes

We first recall in this section the geometrical framework of the packings with fibres parallel to fivefold axes which was analysed in a previous paper (Duneau & Audier, 1999). Simple 2D building rules are then proposed for this type of packing before reporting that two packings of fibres parallel to threefold and fivefold icosahedral axes might be associated.

3.1. Packing of fibres parallel to fivefold axes

From one of the six fivefold axes A_5, we specify the set of fibres parallel to A_5 by their intersections with the orthogonal plane P_5. These positions will be assumed to fall in a $\mathbf{Z}$ module as in the case of the Penrose tilings. More precisely, the fibres are located at points $\mathbf{x} = \sum m_i \boldsymbol{\alpha}_i$, where $m_i \in \mathbf{Z}$ and the five vectors $\boldsymbol{\alpha}_i$ build a regular pentagon of radius $\alpha = |\boldsymbol{\alpha}_i|$. Since $\sum \boldsymbol{\alpha}_i = 0$, the sum of the five integers m_i can be taken in $\{0, 1, 2, 3, 4\}$. The $\mathbf{Z}$ module L generated by $\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_5\}$ contains therefore five subsets $L_k = \{\mathbf{x} \in L; \sum m_i = \mathrm{k}\}$. The subsets L_k ($k = 2, \ldots, 5$) are translations of the submodule L_0 of index 5 in L.

The orbit of a fibre located at $\mathbf{x}$ in the plane P_5 contains ten fibres parallel to A_5 at points $\mathbf{x}_1 = \mathbf{x}, \ldots, \mathbf{x}_{10}$. This number is reduced to five if $\mathbf{x}$ falls on a twofold axis of the plane. The orbit contains five other sets of fibres parallel to the other fivefold axes of the icosahedral group. The projection of the oblique fibres in the plane P_5 gives ten (or five, respectively) stars centred on the points $-\mathbf{x}_i$. If two fibres are located at $\mathbf{x}$ and $\mathbf{y}$, the distances between fibres of the two corresponding orbits can be related to the expression

$$\delta_i = (\mathbf{x} + \mathbf{y}) \cdot \boldsymbol{\alpha}_i / \alpha \quad (i = 1, \ldots, 5),$$

which involves scalar products such as $\mathbf{x} \cdot \boldsymbol{\alpha}_1 = [\Phi(\mathbf{x}) + \Psi(\mathbf{x})/\tau]\alpha^2/2$, where $\Phi(\mathbf{x}) = 2m_1 - m_3 - m_4$ and $\Psi(\mathbf{x}) = m_2 + m_5 - m_3 - m_4$.

For instance, if $\mathbf{y} = \sum n_i \boldsymbol{\alpha}_i$, we obtain

$$\delta_1 = [\Phi(\mathbf{z}) + \Psi(\mathbf{z})/\tau]\alpha/2$$

with $\mathbf{z} = \mathbf{x} + \mathbf{y} = \sum p_i \boldsymbol{\alpha}_i$ and $p_i = m_i + n_i$.

Similar expressions for $\mathbf{x} \cdot \boldsymbol{\alpha}_i$, δ_i ($i = 2, \ldots, 5$) are easily obtained by permutation of indices. Note that the condition $\mathbf{x} \in L_k$ is equivalent to $3\Phi(\mathbf{x}) + \Psi(\mathbf{x}) = k \pmod 5$. The distances δ_i can be lower bounded by conditions of the form $\gamma < \Psi(\mathbf{z}) - \Phi(\mathbf{z})/\tau < \Gamma$, bearing on the conjugate expressions, where the constants γ and Γ have to be specified. Actually, $\Phi(\mathbf{z}) + \Psi(\mathbf{z})/\tau$ and $\Psi(\mathbf{z}) - \Phi(\mathbf{z})/\tau$ can be related to the parallel and perpendicular projections of a 2D lattice point $[\Phi(\mathbf{z}), \Psi(\mathbf{z})]$ on lines of slope τ^{-1} and τ.

An important point is that one has first to find solutions for points with a given value of $\sum m_i$, *i.e.* in a given subset L_k of L. The case $k = 0$ can be rejected at once since for any reasonable window points $\mathbf{x}$ and $-\mathbf{x}$ could both be selected, thereby implying intersections of fibres. The cases $k = 1$ and $k = 4$ are related by an inversion, as well as the cases $k = 2$ and $k = 3$. For a given value of k, points $\mathbf{x}$ and $\mathbf{y}$ belong to L_k and their sum $\mathbf{x} + \mathbf{y}$ belongs to L_{2k}. It follows that the integers $\Phi(\mathbf{z})$ and $\Psi(\mathbf{z})$ verify $3\Phi(\mathbf{z}) + \Psi(\mathbf{z}) = 2k \pmod 5$ so that the 2D lattice points $[\Phi(\mathbf{z}), \Psi(\mathbf{z})]$ run over a sublattice of index 5 in $\mathbf{Z}^2$. The relationship between the minimal distances δ and the associated bounds γ and Γ is easily obtained by application of the cut algorithm in the 2D plane. Finally, different solutions are given by a minimal distance δ and the corresponding acceptance domain defined by conditions $\gamma/2 < \Psi(\mathbf{x}) - \Phi(\mathbf{x})/\tau < \Gamma/2$ ($i = 1, \ldots, 5$). In each case, the density of fibres and the relative volume they fill up can be calculated.

As reported by Duneau & Audier (1999), the best solutions were obtained for a minimal distance $\delta = (2 - \tau)\alpha/2$ in the L_1 ($k = 1$) sublattice. The corresponding bounds are $\gamma = -(8 + 5/\tau)/2 = -\tau^5/2$ and $\Gamma = (3 + 2/\tau)/2 = \tau^3/2$. The acceptance domain W_1 is finally specified by the five conditions $\Psi(\mathbf{x}) - \Phi(\mathbf{x})/\tau < \tau^3/4$. W_1 is a pentagon of radius $\tau^2\alpha/2$.

The same minimal distance can be realized with points in the L_2 ($k = 2$) sublattice. The bounds are $\gamma = -(5 + 3/\tau)/2 = -\tau^4/2$ and $\Gamma = (2 + 1/\tau)/2 = \tau^2/2$. The acceptance domain W_2 is specified by the five conditions $\Psi(\mathbf{x}) - \Phi(\mathbf{x})/\tau < \tau^2/4$ and W_2 is a pentagon of radius $\tau\alpha/2$, τ times smaller than W_1.

It was proved by Duneau & Audier (1999) that the two structures defined by W_1 and W_2 can be combined without inducing intersections of fibres, nor decreasing the minimal distance between them. The relative volume filled by this packing is given by $3(|W_1| + |W_2|)/(50 \times 5^{1/2})\pi\tau^{-4} \approx 0.181\,291$, and the contribution of the L_1 sublattice is $3|W_1|/(50 \times 5^{1/2}) \times \pi\tau^{-4} \approx 0.131\,183$.

Let us note that these constructions were made under the condition of using a common fibre diameter. Now, to consider the possibility of threading fibres of different diameters, there is a solution similar to that presented above for the packing of fibres parallel to the threefold icosahedral axes. In effect, for the first group of fibres characterized by the window W_1 (see Fig. 6*a*), there are two types of holes, the sizes of which are larger than the fibre diameter. The lattice associated with the largest holes corresponds to a self-similar subset of vertical fibres through a τ^3 inflation and a rotation of $\pm(2k+1)\pi/5$ (k is an integer). Therefore, it can easily be deduced from Fig. 6(*a*) that fibres of diameter τ times larger can be threaded through such holes and according to the same icosahedral symmetry. These fibres cannot be in contact with one another but only with the five subsets of pink fibres.

Again in this case, the 2D quasilattice can be determined from a covering cluster. The smallest cluster allowing one to cover all the red nodes characterized by the window W_1 consists of 15 green discs (Fig. 6*b*). Then a large cluster centred on a blue disc and composed of five small clusters can be in turn used to cover both red- and blue-node patterns related by the $-\tau^3$ inflation (Figs. 6*c* and 6*d*). On account of the fivefold symmetry, the beginning of the construction requires the arrangement of at least five large clusters centred at the vertices of a pentagon, such that five green discs define the vertices of a small pentagon in the middle of the previous one, with a π rotation and a size

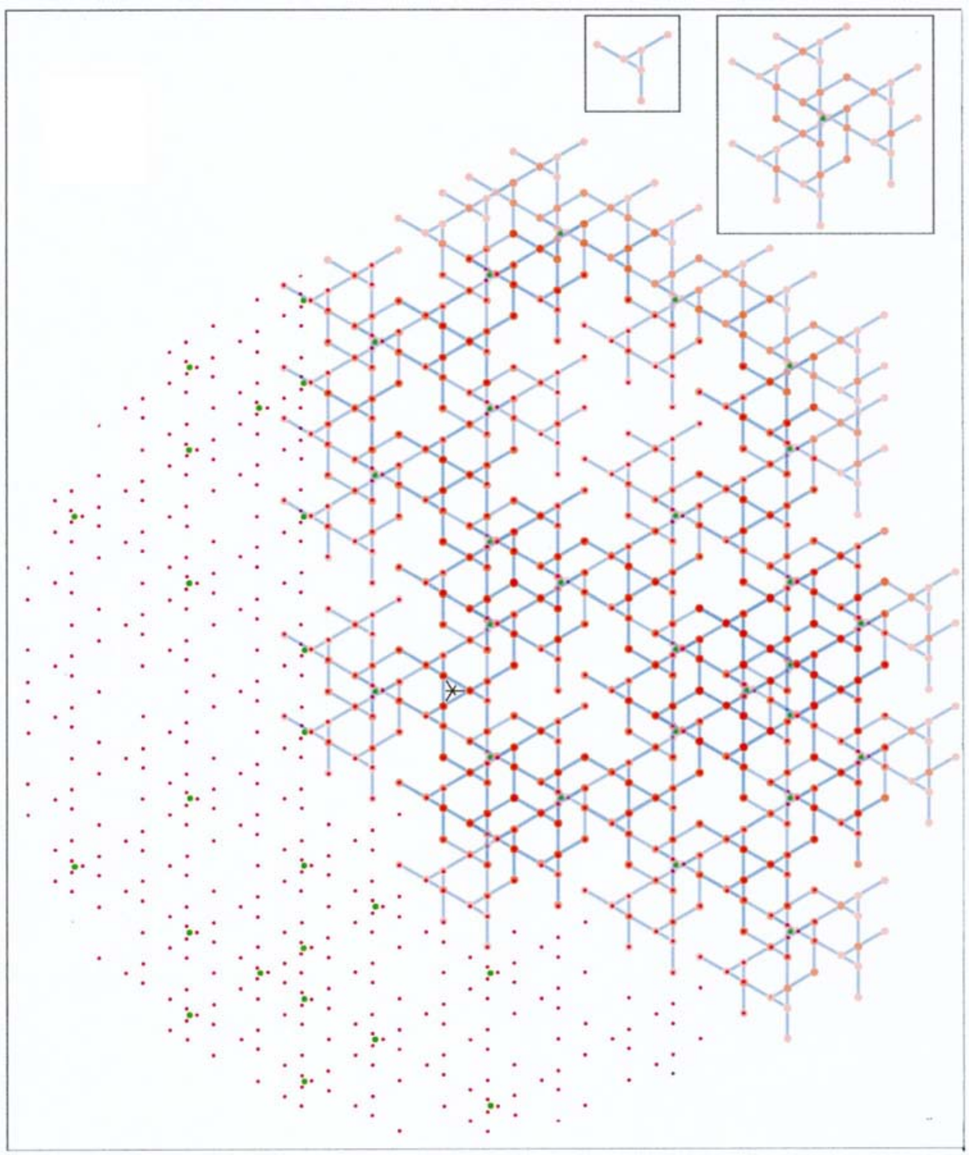


Figure 5
Covering clusters allowing the determination of the 2D pattern corresponding to each subset of fibres parallel to a threefold icosahedral axis and to each related $-\tau^3$-inflated subset. In the inserts, the construction of the largest cluster results from an arrangement of six small clusters partially overlapped and surrounding a disc of violet colour. The construction of the 2D pattern using the largest cluster, as explained in the text (§2.2.5), is here shown by superimposition to correspond to a 2D pattern of nodes, which was previously determined using the cut-and-project method of a periodic hyperspace.

ratio of τ^3. There is only one solution for this first step and through a $-\tau^3$ inflation of all the positions occupied by green discs, a new set of blue nodes are defined where large clusters have to be centred. Such a procedure has to be repeated recurrently.

3.2. Combination of fibres parallel to fivefold and threefold icosahedral axes

The positions of fibres parallel to the fivefold axes A_5 were specified in the orthogonal plane P_5. The vectors $\boldsymbol{\alpha}_i$ are parallel

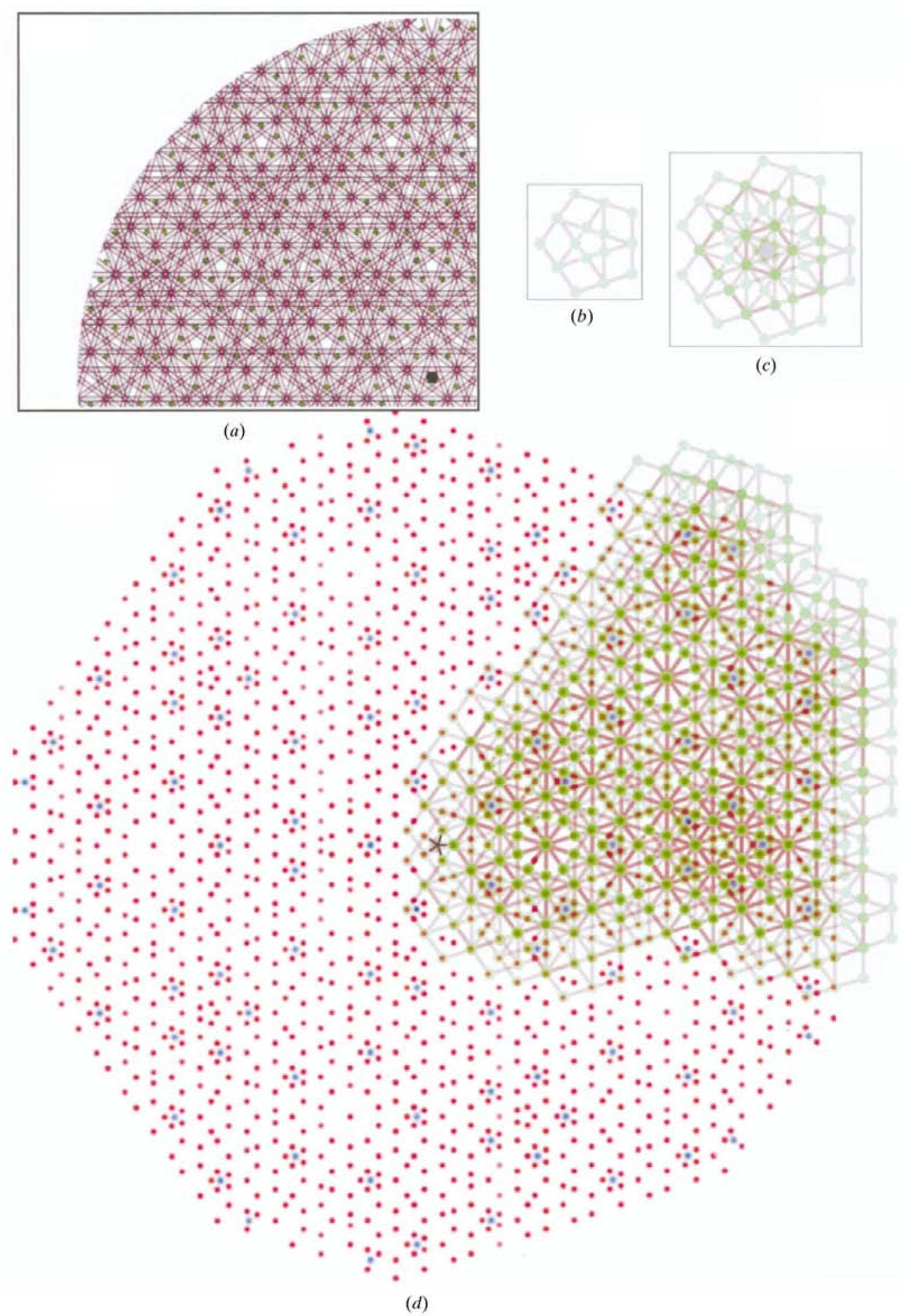
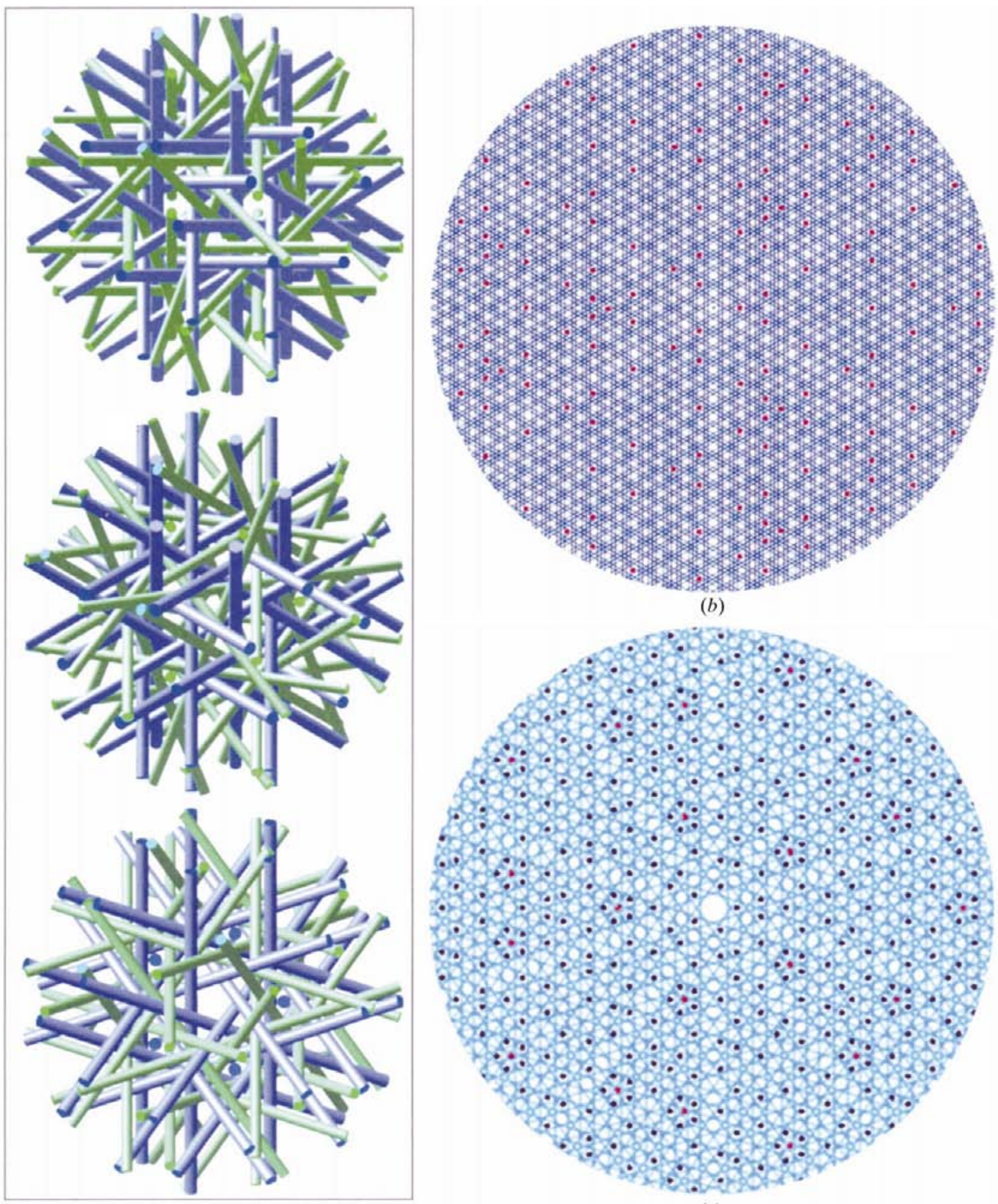


Figure 6
(a) Copy from Duneau & Audier (1999) of the fivefold projection of a packing of fibres parallel to fivefold axes and from which a set of nodes corresponding to vertical fibres are considered as red nodes on the 2D patterns shown below. (b) to (d) Covering clusters in the case of the 2D pattern corresponding to each subset of fibres parallel to a fivefold icosahedral axis and to each related $-\tau^3$-inflated subset (case similar to that presented in Fig. 5).

to twofold axes lying in P_5. A rotation about $\boldsymbol{\alpha}_1$ by $\sim$37.38° transforms A_5 into a threefold axis A_3 perpendicular to a plane P_3. We can identify P_3 with the plane used for the indexing of fibres parallel to threefold axes and we shall assume that $\boldsymbol{\alpha}_1$ and $\mathbf{a}_1$ are parallel. The projections of the fivefold axes into this plane generate only three directions, which are parallel to $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$, respectively. The projections of the $A5$ fibres parallel to A_5 give a set S_5 of lines parallel to $\mathbf{c}_1$; the other projections can be deduced by $2\pi/3$ and $4\pi/3$ rotations. The set S_5 is completely specified by the intersections of the lines with the $\boldsymbol{\alpha}_1$ axis. These points are given by $\xi_1 = \boldsymbol{\xi}\cdot\boldsymbol{\alpha}_1/\alpha = [\Phi(\xi) + \Psi(\xi)/\tau]\alpha/2$ when $\boldsymbol{\xi}$ runs over all allowed positions in P_5. The pentagonal window W_1 selects points $\boldsymbol{\xi}$ such that $3\Phi(\xi) + \Psi(\xi) = 1\,(\mathrm{mod}\,5)$ and $-\tau^2/2 < \Psi(\xi) - \Phi(\xi)/\tau < \tau^3/4$. Similarly, the pentagonal window W_2 selects points $\boldsymbol{\xi}$ such that $3\Phi(\xi) + \Psi(\xi) = 2\,(\mathrm{mod}\,5)$, which satisfies the condition $-\tau/2 < \Psi(\xi) - \Phi(\xi)/\tau < \tau^2/4$. The positions of the lines of S_5 are thus given by a simple 2D cut-and-project algorithm.



(*a*) (*b*) (*c*)

Figure 7

Quasiperiodic packing of fibres parallel to fivefold and threefold icosahedral axes: (*a*) first step of the construction constituted by two subsets of 30 fibres parallel either to the six fivefold axes (in blue) or to the ten threefold axes; (*b*) subset of vertical fibres parallel to a threefold axis (the red points) through holes exhibited by the packing of fibres parallel to the six fivefold axes; (*c*) subset of vertical fibres parallel to a fivefold axis (black points) through holes exhibited by the packing of fibres parallel to the ten threefold axes. [Note: in this last case, a $-\tau^3$-inflated subset of fibres of τ-times-smaller diameter can also be added (red points)].

Table 1
Binary combinations of icosahedral quasiperiodic packings of fibres of circular cross section (see text).

	τP_5 (L)	τP_5 (D)	$\tau^3 P_5$ (L)	$\tau^3 P_5$ (R)	$\tau^0 P_3$ (L) $Rv{\approx}0.0648$	$\tau^0 P_3$ (R) $Rv{\approx}0.0648$	$\tau^3 P_3$ (L)	$\tau^3 P_3$ (R)
$\tau^0 P_5$ (L) $Rv{\approx}0.1312$		$R\varphi = 1$ $Rv{\approx}0.1813$		$R\varphi = \tau$ $Rv{\approx}0.1503$	$R\varphi{\approx}1.1726$ $Rv{\approx}0.1959$	$R\varphi{\approx}1.1726$ $Rv{\approx}0.1959$		
$\tau^0 P_5$ (R) $Rv{\approx}0.1312$	$R\varphi = 1$ $Rv{\approx}0.1813$		$R\varphi = \tau$ $Rv{\approx}0.1503$		$R\varphi{\approx}1.1726$ $Rv{\approx}0.1959$	$R\varphi{\approx}1.1726$ $Rv{\approx}0.1959$		
$\tau^0 P_3$ (L) $Rv{\approx}0.0648$								$R\varphi = \tau$ $Rv{\approx}0.0743$
$\tau^0 P_3$ (R) $Rv{\approx}0.0648$							$R\varphi = \tau$ $Rv{\approx}0.0743$	

We show now that the $A5$ fibres defined by the window W_1 are compatible with the $A3$ fibres constructed previously, once a convenient scaling $K = (\tau + 2)/3 \approx 1.206$ is applied to the $A3$ packing. This scaling is devised to set the relationship $a = K\alpha$ between the norms $|\mathbf{a}_1|$ and $|\boldsymbol{\alpha}_1|$. We first notice that, thanks to the icosahedral symmetry of both packings, we have merely to find a lower bound of the distances between fibres parallel to A_3 and the set S_5. This amounts to a consideration of the differences $\Delta = [\Phi(\boldsymbol{\xi}) + \Psi(\boldsymbol{\xi})/\tau]\alpha/2 - [F(\mathbf{x}) + G(\mathbf{x})/\tau]K\alpha/2$, where $\mathbf{x}$ is the position of an $A3$ fibre. It was shown that $F(\mathbf{x})$ and $G(\mathbf{x})$ satisfy the conditions $-\tau^3 < G(\mathbf{x}) - F(\mathbf{x})/\tau < \tau^3/2$ and $F(\mathbf{x}) = -1 \pmod 3$, $G(\mathbf{x}) = 0 \pmod 3$, which characterize $L_{P,1,0}$. We define now $U = \Phi(\boldsymbol{\xi}) - F(\mathbf{x}) - G(\mathbf{x})/3$ and $V = \Psi(\boldsymbol{\xi}) - [F(\mathbf{x}) + 2G(\mathbf{x}) + 1]/3$. We can check that U and V are integers that satisfy $3U + V = 4 \pmod 5$ and we obtain $\Delta = [U + V/\tau + 1/(3\tau)]\alpha/2$. The conjugate expression $V - U/\tau$ can be shown to fall in the interval $(-\tau^2, 3\tau/2 + 1/4)$. Therefore the values of Δ are given by a 2D cut-and-project algorithm, where the 2D lattice point (U, V) runs over a sublattice of index 5 in $\mathbf{Z}^2$. This allows one to check that $|\Delta| = \Delta_0$, where the lower bound is $\Delta_0 = \tau^{-1}\alpha/3 \approx 0.206\,011\alpha$.

The radius of an $A5$ fibre is $(2 - \tau)a/4$ and that of a (scaled) $A3$ fibre is $(2 - \tau)K\alpha/(2_{1/2}4)$. We notice that their sum ($\approx 0.161\,981\alpha$) is strictly less that Δ_0 so that we can conclude that there is no contact between the two sets of fibres. The relative volume filled by the combination of the two packings is the sum $3|W_1|/(50 \times 5^{1/2})\pi\tau^{-4} + \pi\tau^4/(3^{1/2}192) \approx 0.195\,933$, which is larger than the relative volume of the complete $A5$ packing mentioned above.

As a first step for a construction, two sets of 30 fibres have to be associated. Fig. 7(a) shows such a grouping along twofold, threefold and fivefold directions, where the green and blue fibres are respectively parallel to the ten threefold and six fivefold icosahedral axes. However, it has to be noted that four different constructions are actually possible, as both types of enantiomorphic packings for each subset of 30 fibres parallel to either threefold or fivefold axes can be associated (see §4). Fig. 7(b) shows one of the two possibilities for threading a subset of vertical fibres that are parallel to a threefold axis (the red points) through holes presented by the packing of fibres parallel to the six fivefold axes and characterized by the window W_1. With the same scaling, Fig. 7(c) shows the reverse situation, *i.e.* one of the two possibilities for threading a subset of vertical fibres parallel to a fivefold axis (the black points) through holes exhibited by the packing of fibres parallel to the ten threefold axes. In this second case, another group of fibres of τ-times-smaller diameter can also be added through holes of either a $-\tau^3$-inflated 2D pattern [the red points in Fig. 7(c)] or a $-\tau$-inflated 2D pattern, as described by Duneau & Audier (1999).

4. Discussion

The present description of an icosahedral quasiperiodic packing of fibres parallel to the six fivefold and ten threefold axes results from a combination of two icosahedral closest-packing subsets. In this case, however, fibres of two different diameters must be used and the two subsets are not in contact. It might be of interest to determine, in a further study, if one or several similar configurations could be obtained for fibres of the same circular cross section and/or with contacts between both icosahedral subsets.

Several icosahedral quasiperiodic packings can be built with fibres of different diameters. By way of example, different combinations are gathered in Table 1, though only for fibre diameter ratios equal to 1, $2^{1/2}3/(\tau + 2)$ or τ. Each combination, defined at the crossing between a line and a column, is related to two packing subsets indicated in the first line and first column of the table. Each packing subset is named by a group of symbols: τ^n for the inflation, P_i for a packing with fibre axes parallel to the i-fold directions and L or R for left- or right-handed screw enantiomorphic packing. A relative fibre volume fraction (Rv) is indicated when a packing can be built. The fibre diameter ratio is also indicated for each binary combination. From such a table, it appears that R and L enantiomorphic packings can only be associated when the fibre axes of two subsets remain parallel and that all the combinations (R–L, L–R, L–L and R–R) are possible when the fibre axes are parallel to fivefold and threefold axes. Therefore, one might easily conjecture that mechanical properties of composite materials made with such fibrous reinforcements depend on these different combinations of enantiomorphic structures, in particular a different behaviour depending on the sense of a pure shear. *A priori*, for two opposite senses of a pure shear, the sign of the Poisson coefficient between both lateral strain and longitudinal strain should change.

As a next step in this study, fabrication of different composite samples with periodic and quasiperiodic fibre reinforcement has been undertaken in order to compare their mechanical properties.

References

Duneau, M. (1994*a*). *Proceedings of the 5th International Conference on Quasicrystals*, edited by C. Janot & R. Mosseri, pp. 116–119. Singapore: World Scientific.

Duneau, M. (1994*b*). *Lectures on Quasicrystals, Winter School Aussois*, edited by F. Hippert & D. Gratias, pp. 153–186. Paris: Les Editions de Physique.

Duneau, M. & Audier, M. (1999). *Acta Cryst.* A**55**, 746–754.

Gummelt, P. (1994). *Proceedings of the 5th International Conference on Quasicrystals*, edited by C. Janot & R. Mosseri, pp. 84–87. Singapore: World Scientific.

Janot, C. (1994). *Quasicrystals, a Primer.* Oxford University Press.

Parkhouse, J. G. & Kelly, A. (1998). *Proc. R. Soc. London Ser. A*, **454**, 1889–1909.